Close today: HW_3D, 4A,4B (6.2/6.3,6.4)
Close next Wed: HW_5A, 5B, 5C (6.5,7.1,7.2)

6.5 Average Value

The average *y*-value of $y = f(x)$ from $x = a$ to $x = b$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Entry Task: The formula for the temperature of a particular object is $T(t) = t^2$ degrees Fahrenheit where t is in hours. Find the average temperature from $t = 1$ to $t = 4$ hours.

The mean value theorem for integrals:

If $f(x)$ is continuous on from $x = a$ to $x = b$, then there is at least one value $x = c$ at which

$$f(c) = f_{ave}.$$

Example:

Using $T(t) = t^2$ from $t = 1$ to $t = 4$ again. Find a time at which the temperature is exactly equal to the average value.

Average Value Derivation

The average value of the *n* numbers:

$y_1, y_2, y_3, \ldots, y_n$

is given by

$$\frac{y_1+y_2+y_3+\cdots+y_n}{n} = y_1\frac{1}{n} + \cdots + y_n\frac{1}{n}.$$

Goal: We want the average value of **all** the y-values of some function y = f(x) over an interval x = a to x = b.

Derivation:

1. Break into *n* equal subdivisions

$$\Delta x = \frac{b-a}{n}, \text{ which means } \frac{\Delta x}{b-a} = \frac{1}{n}$$

2. Compute *y*-value at each tick mark

$$y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$$

3. Ave $\approx f(x_1)\frac{\Delta x}{b-a} + \cdots + f(x_n)\frac{\Delta x}{b-a}$

$$\text{Average} \approx \frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x$$

Thus,

$$\text{Average} = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

7.1 Integration by Parts

Goal: We will reverse the product rule.

Before we start, add these to your basic list of integrals:

$$
\begin{aligned}
\int \sin(ax+b)\,dx &= -\frac{1}{a}\cos(ax+b) + C \\
\int \cos(ax+b)\,dx &= \frac{1}{a}\sin(ax+b) + C \\
\int e^{ax+b}\,dx &= \frac{1}{a}e^{ax+b} + C \\
\int \frac{1}{\mathrm{ax}+\mathrm{b}}\,dx &= \frac{1}{a}\ln|ax+b| + C
\end{aligned}
$$

Derivation of Integration By Parts

The <u>product rule</u> says:

$$u(x)v'(x) + v(x)u'(x) = \frac{d}{dx}(u(x)v(x))$$

which can be written as

$$\int u(x)v'(x)dx + \int v(x)u'(x)dx = u(x)v(x)$$

Writing this in terms of the differentials:

$$dv = v'(x)dx \text{ and } du = u'(x)dx$$

we have

$$\int u\,dv + \int v\,du = uv$$

which we rearrange to get

Integration by Parts formula:

$$\int u\,dv = uv - \int v\,du$$

Example:

$$\int x\cos(8x)dx$$

Step 1: Choose *u* and *dv*.

Step 2: Compute *du* and *v*.

Step 3: Use formula (and hope)

Example:

$$\int x^2 \ln(x)\, dx$$

Example:

$$\int_{1}^{e} x^2 \ln(x)\, dx$$

Notes:

1. The symbols *u* and *v* **never** appear in the integration. They are just locations in the formula (no variables are changing, this is not substitution).

2. *u* and *dv* completely split up the integrand. **Once you <u>chose *u*</u>, then *dv* is everything else**.

3. The goal is to make $\int v\,du$ "nicer" than $\int u\,dv$
   (a) Pick u = "something that gives a derivative that is simpler than the original u"
   (b) Pick dv = "something that you can integrate"
   (c) And hope "vdu" is something in our table!

Example:

$$\int x^2 e^{x/2} dx$$

Example:

$$\int e^x \cos(x)\, dx$$

Example:

$$\int \sin^{-1}(x)\, dx$$